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QUASI-FACTORS IN ERGODIC THEORY

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ABSTRACT
Motivated by the notion of quasi-factor in topological dynamics, we introduce an analogous notion in the context of ergodic theory. For two processes, $\mathscr{X}$ and $\mathscr{Y}$, we have $\mathscr{X} \not\perp \mathscr{Y}$ if and only if $\mathscr{Y}$ has a factor which is isomorphic to a quasi-factor of $\mathscr{X}$. On the other hand, weakly mixing processes can have nontrivial quasifactors which are not w.m. We characterize those ergodic processes which admit only trivial continuous ergodic quasi-factors, and use this characterization to conclude that a process with minimal selfjoinings is of this type. From this we derive the fact that for every such $\mathscr{X}$ and any ergodic $\mathscr{Y}$ either $\mathscr{X} \perp \mathscr{Y}$ or $\mathscr{Y}$ extends some symmetric product of $\mathscr{X}$.

§1. Quasi-factors

Motivated by the notion of quasi-factor in topological dynamics ([1], [2], [3], [4]), we introduce an analogous notion in the context of ergodic theory as follows. Let $\mathscr{X} = (X, \mathscr{B}, \mu, T)$ be a measure-preserving process where X is a compact metric space, $\mathscr{B}$ its Borel field, μ a probability measure, and T a homeomorphism of X. We call such a process *topological.* Let $\mathscr{P}(X)$ be the space of probability measures on X equipped with the weak $*$ topology, and let $\mathscr{G}$ be the corresponding Borel field. T induces an affine homeomorphism of $\mathscr{P}(X)$ which we denote also by T.

Definition. We say that a process $(\mathscr{P}(X), \mathscr{G}, \lambda, T)$ is a *quasi-factor* of X if $\lambda \in \mathscr{P}(\mathscr{P}(X))$ is (i) T-invariant and (ii) μ is the barycenter of λ, i.e., for every $g \in C(X)$ (continuous functions on X)

$$\int_{\mathscr{P}(X)} \int_X g(x)d\nu(x)d\lambda(\nu) = \int g(x)d\mu(x).$$

Our first goal is to show that a q.f. is an invariant of the (measure theoretical) process $(X, \mathscr{B}, \mu, T)$.

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For the following lemma we found no reference; the proof given was indicated to us by J. Aaronson.

LEMMA 1.1. *Let $\mathscr{X} = (X, \mathscr{B}, \mu, T)$ and $\mathscr{Y} = (Y, \mathscr{F}, \eta, T)$ be two topological processes. Suppose there is a measurable isomorphism between $\mathscr{X}$ and $\mathscr{Y}$, i.e., there exist Borel measurable sets $X_0' \subset X$ and $Y_0' \subset Y$ with $\mu(X_0') = \eta(Y_0') = 1$ and an equivariant* 1-1 *map $X_0' \xrightarrow{\varphi} Y_0'$ such that φ and φ^{-1} are measurable with respect to $\bar{\mathscr{B}}$ and $\bar{\mathscr{F}}$ the completions of $\mathscr{B}$ and $\mathscr{F}$, respectively. Then there exist Borel subsets of measure one, $X_0 \subset X_0'$ and $Y_0 \subset Y_0'$ such that $X_0 \xrightarrow{\varphi} Y_0$ is Borel measurable.*

PROOF. Let $P_n = \{A_{n,1}, A_{n,2}, \cdots, A_{n,k_n}\}$ be a sequence of partitions of Y where for every n, j, $A_{n,j} \in \mathscr{F}$ and diameter $(A_{n,j}) < 1/n$. Let $\tilde{A}_{n,j} = \varphi^{-1} A_{n,j}$; then $A_{n,j}$ is in $\bar{\mathscr{B}}$ and we can choose $B_{n,j} \in \mathscr{B}$ such that $B_{n,j} \subset \tilde{A}_{n,j}$ with $\mu(B_{n,j}) = \mu(\tilde{A}_{n,j})$. Choose $y_{n,j} \in A_{n,j}$ and define $\varphi_n : \bigcup_{j=1}^{k_n} B_{n,j} \to Y$ by $\varphi_n(x) = y_{n,j}$ if $x \in B_{n,j}$. Put $X_0'' = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{k_n} B_{n,j}$; clearly we have $\mu(X_0'') = 1$ and $\lim \varphi_n(x) = \varphi(x)$ for every $x \in X_0''$. Since φ_n is Borel measurable on X_0'', it follows that $\varphi \mid X_0''$ is Borel measurable, and since φ is 1-1, the set $\varphi(X_0'') = Y_0''$ is also Borel. Finally, put $X_0 = \bigcap_{n=-\infty}^{\infty} T^n X_0''$, $Y_0 = \varphi(X_0)$. □

LEMMA 1.2. *Let X be compact metric; then the set $\{\nu : \nu(f) > a\}$ is a Borel subset of $\mathscr{P}(X)$ for every $a \in \mathbf{R}$ and f a bounded Borel function on X.*

PROOF. Let $\mathscr{D}$ be the class of bounded functions f on X which are ν-integrable for every $\nu \in \mathscr{P}(X)$ and such that $\forall a \in \mathbf{R}$, $\{\nu : \nu(f) > a\}$ is a Borel subset of $\mathscr{P}(X)$. By the definition of the topology on $\mathscr{P}(X)$, we have $C(X) \subset \mathscr{D}$. If $f_n \in \mathscr{D}$ and $f = \lim f_n$ (pointwise) is bounded, then $\forall \nu$, $\lim \nu(f_n) = \lim \nu(f)$ and $\forall a$, $\nu(f) > a \Leftrightarrow$ eventually $\nu(f_n) > a$. Thus

$$\{\nu : \nu(f) > a\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{\nu : \nu(f_n) > a\}.$$

Whence $\mathscr{D}$ contains all Borel bounded functions, and the proof is completed. □

PROPOSITION 1.3. *Let $\mathscr{X} = (X, \mathscr{B}, \mu, T)$ and $\mathscr{Y} = (Y, \eta, \mathscr{F}, T)$ be two topological processes. Suppose there is a measurable isomorphism between $\mathscr{X}$ and $\mathscr{Y}$; then to every* q.f. *of $\mathscr{X}$ there corresponds a (measure theoretically) isomorphic* q.f. *of $\mathscr{Y}$.*

PROOF. We observe first that the barycenter equation (ii) holds for every Borel bounded function on X. (The class $\mathscr{D}$ of bounded functions for which it holds contains $C(X)$ and is closed under pointwise limits (bounded), by the Lebesgue bounded convergence theorem used twice.) Since $\mathscr{X}$ and $\mathscr{Y}$ are measurably isomorphic, there exist, by Lemma 1.1, Borel subsets $X_0 \subset X$, $Y_0 \subset Y$

with $\mu(X_0)=1=\eta(Y_0)$ and a 1-1 Borel isomorphism $(X_0,\mathcal{B}_0,\mu,T)\xrightarrow{\varphi}(Y_0,\mathcal{F}_0,\eta,T)$, where $\mathcal{B}_0=\{B\cap X_0 : B\in\mathcal{B}\}$ and $\mathcal{F}_0$ is defined similarly. Let $\mathcal{P}_0(\mathcal{P}'_0)$ be the Borel subsets of $\mathcal{P}(X)(\mathcal{P}(Y))$ which consist of the measures ν with $\nu(X_0)=1$ $(\nu(Y_0)=1)$, respectively (Lemma 1.2). Now define $\varphi_*:\mathcal{P}_0\to\mathcal{P}'_0$ as follows, for $g\in C(Y)$

$$\varphi_*(\nu)(g)=\int(g\circ\varphi)d\nu.$$

Clearly $\varphi_*(\nu): C(Y)\to\mathbf{R}$ is a bounded linear functional. Thus $\varphi_*(\nu)\in\mathcal{P}(Y)$ and moreover $\varphi_*(\nu)\in\mathcal{P}'_0$. Also φ_* is Borel measurable (Lemma 1.2), and T-equivariant. Since $(\varphi_*)^{-1}=(\varphi^{-1})_*$ exists, φ_* is a Borel isomorphism.

Now let $\lambda\in\mathcal{P}(\mathcal{P}(X))$ be a q.f. of X. Computing the integral of 1_{X_0} we have

$$1=\int 1_{X_0}d\mu=\iint 1_{X_0}d\nu d\lambda(\nu).$$

Thus $\nu(X_0)=1$ for λ-a.e. ν and $\lambda(\mathcal{P}_0)=1$. We let $\lambda'=\varphi_{**}(\lambda)\in\mathcal{P}(\mathcal{P}(Y))$, then λ' is an invariant measure supported on $\mathcal{P}'_0$. For $g\in C(Y)$, we have

$$\iint g d\nu' d\lambda'(\nu')=\iint g d\nu' d(\varphi_{**}\lambda)(\nu')=\iint g d(\varphi_*(\nu))d\lambda(\nu)$$

$$=\iint(g\circ\varphi)d\nu d\lambda(\nu)=\int g\circ\varphi d\mu=\int g d\eta.$$

Thus λ' is a q.f. of $\mathcal{Y}$ and

$$\varphi_*:(\mathcal{P}(X),\mathcal{G},\lambda,T)\to(\mathcal{P}(Y),\mathcal{G}',\lambda',T)$$

is an isomorphism of measure-preserving systems. □

We can now define the quasi-factors of any process $\mathcal{X}$ to be the quasi-factors of any topological realization of $\mathcal{X}$.

Proposition 1.4. *Let $\mathcal{X}=(X,\mathcal{B},\mu,T)$ and $\mathcal{Y}=(Y,\mathcal{F},\eta,T)$ be topological processes with $\mathcal{X}\xrightarrow{\varphi}\mathcal{Y}$ a homomorphism. Then $\mathcal{Y}$ is isomorphic to a* q.f. *of $\mathcal{X}$.*

Proof. Let $\mu=\int\mu_y d\eta(y)$ be a disintegration of μ over η w.r.t. φ. Define $\psi: Y\xrightarrow{\varphi}\mathcal{P}(X)$ by $\psi(y)=\mu_y$ and let $\lambda\in\mathcal{P}(\mathcal{P}(X))$ be the measure $\psi_*(\eta)$. Then λ is T-invariant and for every $g\in C(X)$

$$\int_{\mathcal{P}(X)}\int_X g d\nu d\lambda(\nu)=\iint g d\mu_y d\eta(y)=\int g d\mu.$$

Thus λ is a q.f. of $\mathcal{X}$ and since ψ is 1-1, $\mathcal{Y}$ is isomorphic to this q.f. □

PROPOSITION 1.5. *Let $\mathscr{X}=(X,\mathscr{B},\mu,T)$ and $\mathscr{Y}=(Y,\mathscr{F},\eta,T)$ be topological processes. Then $\mathscr{X}$ is not disjoint from $\mathscr{Y}$ iff $\mathscr{Y}$ has a factor which is isomorphic to a non-trivial* q.f. *of $\mathscr{X}$.*

PROOF. $\mathscr{X}\not\perp\mathscr{Y} \Leftrightarrow$ there exists an invariant measure $\vartheta\in\mathscr{P}(X\times Y)$ such that $\pi_X\vartheta=\mu$, $\pi_Y\vartheta=\eta$ and $\vartheta\neq\mu\times\eta$. (We say in this case that the process $(X\times Y,\mathscr{B}\otimes\mathscr{F},\vartheta,T)$ is a joining of $\mathscr{X}$ and $\mathscr{Y}$.)

Let $\vartheta=\int\mu_y\times\delta_y d\eta(y)$ be a disintegration of ϑ over η w.r.t. π_Y. Then the map $\psi:Y\to\mathscr{P}(X)$ defined by $\psi(y)=\mu_y$ can be used to define the q.f. $\lambda=\psi_*(\eta)$ of $\mathscr{X}$. In fact, for $g\in C(X)$

$$\int_{\mathscr{P}(X)}\int_X g d\nu d\lambda(\nu)=\int_Y\int_X g d\mu_y d\eta(y)=\int g(x)d\vartheta(x,y)=\int g(x)d\pi_X\vartheta=\int g d\mu.$$

Conversely, if $\psi:(Y,\eta)\to(\mathscr{P}(X),\lambda)$ is a homomorphism of (Y,η) onto the q.f. λ of $\mathscr{X}$, then the measure $\vartheta=\int\psi(y)\times\delta_y d\eta(y)$ is a joining of $\mathscr{X}$ and $\mathscr{Y}$.

In both directions λ is trivial iff $\lambda=\delta_\mu$ iff $\psi(y)=\mu$, η-a.e. iff $\vartheta=\mu\times\eta$. This completes the proof. □

PROPOSITION 1.6. *If $\mathscr{X}$ is a Kronecker ergodic process, then every* q.f. *of $\mathscr{X}$ is isomorphic to a factor of $\mathscr{X}$.*

PROOF. We can assume that X is a compact abelian topological group, $Tx=ax$ where $\{a^n\}_{n\in\mathbb{Z}}$ is a dense subgroup of X, and μ is Haar measure on X.

Let $\lambda\in\mathscr{P}(\mathscr{P}(X))$ be an ergodic q.f. Let $\nu_0\in\mathscr{P}(X)$ be a generic point for λ. Then $\mathrm{Supp}(\lambda)\subset\bar{\mathscr{O}}(\nu_0)$ (orbit closure of ν_0 in $\mathscr{P}(X)$). The action of X on itself induces an action of X on $\mathscr{P}(X)$ and it is clear that $\bar{\mathscr{O}}(\nu_0)=\{x\circ\nu_0:x\in X\}$. Thus $\bar{\mathscr{O}}(\nu_0)$ is a factor group of X and λ is Haar measure on $\bar{\mathscr{O}}(\nu_0)$. □

The last three propositions demonstrated some aspects of the notion of q.f. which make it similar to that of a factor. In the next proposition we will show how far can a q.f. be from being a factor. (See [2] for the topological analogue.)

PROPOSITION 1.7. *Let $\mathscr{X}=(X,\mathscr{B},\mu,T)$ be a weakly mixing topological process. Let K be the circle group. There exists a weakly mixing process $\mathscr{Z}$ on $Z=X\times K\times K$ which extends $\mathscr{X}$ so that $\mathscr{Z}$ possesses a* q.f. *with an eigenvalue* -1.

PROOF. Let $\varphi:X\to K$ be some continuous function such that the transformation $T:X\times K\to X\times K$ defined by $T(x,k)=(Tx,k\varphi(x))$, is weakly mixing. Next define $T:Z\to Z$ by

$$T(x,k,k')=(Tx,k\varphi(x),k'k).$$

We claim that T on Z is weakly mixing. In fact, suppose $f \in L_2(d\eta)$ where $d\eta = d\mu dk dk'$ is an eigenfunction of T with eigenvalue λ. Write $f(x,k,k') = \sum_{n=-\infty}^{\infty} c_n(x,k)k'^n$; then

$$\lambda \sum c_n(x,k)k'^n = \sum c_n(Tx, k\varphi(x))k^n k'^n.$$

Hence for every n, $\lambda c_n(x,k) = k^n c_n(Tx, k\varphi(x))$. For a fixed n, writing $c_n(x,k) = g(x,k)$ we have $\lambda g(x,k) = k^n g(T(x,k))$. Write $g(x,k) = \sum_{j=-\infty}^{\infty} b_j(x)k^j$, then $\lambda \sum b_j(x)k^j = \sum b_j(Tx)\varphi(x)^j k^{j+n}$; and therefore, for every j, $\lambda b_{j+n}(x) = b_j(Tx)\varphi(x)^j$. Since $|\lambda| = |\varphi(x)| = 1$, we have $\int |b_{j+n}(x)|^2 d\mu = \int |b_j(x)|^2 d\mu$. But $\int |g|^2 d\mu dk = \sum \int |b_j|^2 d\mu$. Hence for non-zero g this can hold only for $n = 0$. Thus $f(x,k,k') = c_0(x,k)$ and $\lambda c_0(x,k) = c_0(T(x,k))$. By weak mixing of $X \times K$ we have that f is a constant, and our claim is proved.

Consider now the following correspondences for $z = (x,k,k')$; let $\psi_1(z) = \frac{1}{2}(\delta_{(x,k,k')} + \delta_{(x,-k,k')}) = \nu_z$ and $\psi_2(z) = \frac{1}{2}(\delta_{(x,k,k')} + \delta_{(x,-k,-k')}) = \bar{\nu}_z$. Put

$$\lambda = \tfrac{1}{2}((\psi_1)_*(\eta) + (\psi_2)_*(\eta)).$$

Then it is easy to check that λ is a q.f. of $\mathscr{Z} = (Z, \eta, T)$. Now

$$T\nu_z = \tfrac{1}{2}(\delta_{(Tx, k\varphi(x), k'k)} + \delta_{(Tx, -k\varphi(x), -k'k)}) = \bar{\nu}_{Tz}.$$

Thus the function $F(\nu_z) = 1$, $F(\bar{\nu}_z) = -1$, $\forall z \in Z$, which is defined λ a.e., is an eigenfunction of T with eigenvalue -1. □

Remark. The same procedure can be taken with $K = \{\pm 1\}$, and we conclude that every Bernoulli process has a non-w.m. q.f.

§2. *M*-Processes

Let $\mathscr{X}$ be an ergodic topological process, let $\mathscr{P}_c(X)$ and $\mathscr{P}_d(X)$ be the subspaces of $\mathscr{P}(X)$ consisting of continuous and discrete measures respectively. Let λ be an ergodic q.f. of $\mathscr{X}$ and write $\nu = a(\nu)\nu_c + (1 - a(\nu))\nu_d$, the decomposition of ν into continuous and discrete parts. By ergodicity of λ, $a(\nu) = a$, $0 \leqq a \leqq 1$, λ-a.e. We then have

$$\mu = \int \nu d\lambda(\nu) = a \int \nu_c d\lambda(\nu) + (1-a)\int \nu_d d\lambda(\nu) = a\mu_1 + (1-a)\mu_2.$$

By ergodicity of μ, $\mu_1 = \mu_2 = \mu$. Thus, λ induces two q.f., λ_c and λ_a, supported on $\mathscr{P}_c(X)$ and $\mathscr{P}_d(X)$, respectively. We say that λ is *discretely* (*continuously*) *supported* if $\lambda_c = 0$ ($\lambda_d = 0$).

Next we examine the nature of an ergodic discretely supported q.f. $(\mathcal{P}(X), \lambda, T)$ of an ergodic topological process $\mathcal{X}$. Let $\nu^0 \in \mathcal{P}_d(X)$ be a generic point for λ and write $\nu^0 = \sum_{i=1}^{\infty} \nu_i^0$ where $\nu_i^0 = a_i \sum_{j=1}^{n_i} \delta_{x_{i,j}^0}$ and $\sum_{i=1}^{\infty} n_i a_i = 1$. Let $N = \bar{\mathcal{O}}(\nu^0)$ be the orbit closure of ν^0 in $\mathcal{P}_d(X)$. A typical point of N has the form $\nu = \sum_{i=1}^{\infty} \nu_i$ where $\nu_i = a_i \sum_{j=1}^{n_i} \delta_{x_{i,j}}$. For every i we let

$$X_i = \underbrace{X \times \cdots \times X}_{n_i}$$

and let $X^{\hat{n}_i} = \hat{X}_i$ be the quotient space of X_i under the group S_{n_i} of permutations of the coordinates. We have a natural T-equivariant map φ from N into $\prod_{i=1}^{\infty} \hat{X}_i$. Let $M = \varphi(N)$, $\sigma = \varphi_*(\lambda)$ and for every i let $\pi_i : M \to \hat{X}_i$ be the projection map. For every measure η on $\hat{X}_i$, there is a unique S_{n_i} invariant measure $\tilde{\eta}$ on X_i with $(\pi_i)_* \tilde{\eta} = \eta$. Finally, let $\pi_{i,j}$ be the projection of X_i on its j-th coordinate. When $n_i = 1$, we have of course $\hat{X}_i = X_i$. We now claim that for every i, j, $(\pi_{i,j})_* (\widetilde{(\pi_i)_*\sigma}) = \mu$. (In particular when $n_i = 1$, $(\pi_i)_*\sigma = \mu$.) In fact, we have for every $f \in C(X)$

$$\begin{aligned} \int f d\mu &= \int f(\nu) d\lambda(\nu) = \int f\left(\sum_{i=1}^{\infty} \nu_i\right) d\lambda(\nu) = \int \sum_{i=1}^{\infty} f(\nu_i) d\lambda(\nu) \\ &= \sum_{i=1}^{\infty} \int f(\nu_i) d\lambda(\nu) = \sum_{i=1}^{\infty} \int a_i \sum_{j=1}^{n_i} f(x_{i,j}) d(\widetilde{(\pi_i)_*\sigma}) \\ &= \sum_{i=1}^{\infty} a_i \sum_{j=1}^{n_i} \int f(x_{i,j}) d(\pi_{i,j})_* (\widetilde{(\pi_i)_*\sigma})(x_i) \end{aligned}$$

and the ergodicity of μ implies $\mu = (\pi_{ij})_* (\widetilde{(\pi_i)_*\sigma})$, $\forall i, j$.

Let $\vartheta \in \mathcal{P}(X \times Y)$ be a joining of $\mathcal{X}$ and $\mathcal{Y}$ which is ergodic on $X \times Y$. Let $\vartheta = \int \mu_y \times \delta_y d\eta(y)$ be its disintegration over η and suppose that the q.f. $\lambda = \psi_*(\eta)$, as in the proof of Proposition 1.3, is discretely supported and ergodic. As above, let ν^0 be a generic point for λ and $\nu(y) = \sum_{i=1}^{\infty} \nu_i(y) = \mu_y$ a typical point of the process $(\mathcal{P}(X), \lambda, T)$. Then

$$\vartheta = \int \nu(y) \times \delta_y d\eta(y) = \int \sum \nu_i(y) \times \delta_y d\eta(y) = \sum \int \nu_i(y) \times \delta_y d\eta(y) = \sum \vartheta_i.$$

The ϑ_i are clearly invariant measures on $X \times Y$, and if more than one α_i is different from zero, this contradicts the ergodicity of ϑ. Thus there exists i_0 with $a_{i_0} = 1/n_0$ and for all i, $i \neq i_0$, $a_i = 0$; i.e., $\nu = (1/n_0) \sum_{j=1}^{n_0} \delta_{x_j}$. We proved the following:

PROPOSITION 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological processes, $\vartheta \in \mathcal{P}(X \times Y)$ an ergodic joining and $(\mathcal{P}(X), \lambda, T)$, where $\lambda = \psi_*(\eta)$, the corresponding* q.f. *as in*

Proposition 1.3. *Suppose λ is discretely supported; then there exist a positive integer n_0 and a measure σ on the symmetrized product $X^{\hat{n}_0}$ with $(\pi_i)_*(\tilde{\sigma}) = \mu$, $1 \leqq i \leqq n_0$ such that $(X^{\hat{n}_0}, \sigma, T)$ is isomorphic to $(\mathscr{P}(X), \lambda, T)$.*

DEFINITION. Let $\mathscr{X} = (X, \mathscr{B}, \mu, T)$ be a topological process. We say that a measurable subset $F \subset X \times X$ is *sparse* if for every selfjoining on $X \times X$, i.e., every invariant probability measure ϑ on $X \times X$ with $(\pi_i)_* \vartheta = \mu$ $(i = 1,2)$, such that $\vartheta(F) = 1$, we have in the disintegration $\vartheta = \int \delta_x \times \vartheta_x d\mu(x)$ of ϑ over μ, that μ-almost every ϑ_x is discrete. We say that $\mathscr{X}$ is an *M-process* if the set $F = \{(x, y) \in X \times X : (x, y)$ is not generic for $\mu \times \mu\}$ is sparse.

THEOREM 2.2. *The following conditions on an ergodic topological process $\mathscr{X} = (X, \mathscr{B}, \mu, T)$ are equivalent.*

(i) *$\mathscr{X}$ is an M-process.*

(ii) *Every ergodic continuously supported* q.f. $(\mathscr{P}(X), \lambda, T)$ *is trivial* (i.e., $\lambda = \delta_\mu$).

(iii) *For every ergodic topological process $\mathscr{Y} = (Y, \mathscr{F}, \eta, T)$ and every joining $\vartheta \in \mathscr{P}(X \times Y)$, with $\vartheta \not\geqslant \mu \times \eta$, the measures μ_y in the disintegration $\vartheta = \int \mu_y \times \delta_y d\eta(y)$ have discrete support for η* a.e. y.

PROOF. (i) $\Rightarrow$ (ii): Let $(\mathscr{P}(X), \lambda, T)$ be a continuously supported ergodic q.f. of $\mathscr{X}$. Put

$$\vartheta = \int (\nu \times \nu) d\lambda(\nu).$$

Then ϑ is a selfjoining on $X \times X$; let $F = \{(x, y) \in X \times X : (x, y)$ is not generic for $\mu \times \mu\}$ and assume $\vartheta(F) > 0$. Since F is an invariant set, the restriction $\tilde{\vartheta}$ of ϑ to F, normalized to be a probability measure, clearly is an invariant measure with $(\pi_i)_* \tilde{\vartheta} = \mu$ $(i = 1,2)$ (μ is ergodic) and $\tilde{\vartheta}(F) = 1$. By our assumption then,

$$\tilde{\vartheta} = \int \delta_x \times \vartheta_x d\mu(x)$$

where ϑ_x is discrete for every $x \in A \subset X$ with $\mu(A) = 1$. Let $F_1 \subset F$ be a measurable set with $\pi_1(F_1) = A$, such that $\{y : (x, y) \in F_1\} = F_{1,x}$ is countable for each $x \in A$ and $\tilde{\vartheta}(F_1) = 1$. Then

$$\begin{aligned} 1 = \tilde{\vartheta}(F_1) &= \int_{\mathscr{P}(X)} \int_X \int_X 1_{F_1}(x, y) d\nu(x) d\nu(y) d\lambda(\nu) \\ &= \int_{\mathscr{P}(X)} \int_A \left[\int_X 1_{F_{1,x}}(y) d\nu(y) \right] d\nu(x) d\lambda(\nu). \end{aligned}$$

But $\int_X 1_{F_{1,x}}(y)d\nu(y) = 0$ for every $x \in A$ and λ almost every ν, since $F_{1,x}$ is countable for all $x \in A$ and λ-almost every ν is a continuous measure, this contradiction shows that $\vartheta(F) = 0$. Since $\vartheta(F) = \int\int 1_F d\nu \times \nu d\lambda(\nu)$ we conclude that for λ-almost every ν, $\nu \times \nu(F) = 0$. Now if $(x, y) \notin F$, then for an arbitrary continuous function f on X we have

$$\frac{1}{2N+1}\sum_{n=-N}^{N} f(T^n x)\bar{f}(T^n y) \to \left|\int f(\xi)d\mu(\xi)\right|^2.$$

In particular, for λ a.e. ν this convergence holds $\nu \times \nu$-a.e. For such ν we therefore have

$$(*) \qquad \frac{1}{2N+1}\sum_{n=-N}^{N}\left|\int f(T^n x)d\nu(x)\right|^2 \to \left|\int f(x)d\mu(x)\right|^2.$$

For f with $\int f d\mu = 0$, we conclude the existence of a subset J_f of $\mathbf{Z}$ of density 1 for which $\lim_{n \in J_f} \int f dT^n \nu \to 0$. Let $\{f_i\}_{i=1}^{\infty}$ be a dense subset of $C_0(X) = \{f \in C(X) : \int f d\mu = 0\}$; then for λ a.e. ν, the convergence in $(*)$ holds for every $f \in \{f_i\}_{i=1}^{\infty}$.

In particular, we can find ν_0 which is a generic point for λ for which this is true. Choose $J_i = J_{f_i}$ as above and let $J \subset \mathbf{Z}$ be of upper density 1 such that $\forall i \, \exists N_i$ with $J \cap [-N_i, N_i]^c \subset J_i \cap [-N_i, N_i]^c$, then we have

$$\lim_{n \to \infty, n \in J} \int f dT^n \nu_0 = \int f d\mu \qquad \text{for every } f \in C(X),$$

i.e.,

$$\lim_{n \to \infty, n \in J} T^n \nu_0 = \mu.$$

This clearly implies $\lambda = \delta_\mu$ and the proof of (i) $\Rightarrow$ (ii) is completed.

(ii) $\Rightarrow$ (iii): Let $\vartheta \in \mathcal{P}(X \times Y)$, $\vartheta \not\gg \mu \times \eta$ be a joining of $\mathcal{X}$ and $\mathcal{Y}$, $\vartheta = \int \mu_y \times \delta_y d\eta(y)$ and $(\mathcal{P}(X), \lambda, T)$ the q.f. of $\mathcal{X}$ where $\lambda = \psi_*(\eta)$ as in the proof of Proposition 1.3. Now (ii) implies that $\lambda_c = 0$ or $\lambda_c = \delta_\mu$. The latter can occur only when $\mu_y \gg \mu$ for ν-a.e. y iff $\vartheta \gg \mu \times \eta$. Thus $\lambda_c = 0$, and μ_y is discrete for η-a.e. y.

(iii) $\Rightarrow$ (i): Assume $\mathcal{X}$ satisfies (iii) and $F = \{(x, y) \in X \times X : (x, y)$ is not ergodic for $\mu \times \mu\}$. Let $\vartheta \in \mathcal{P}(X \times X)$ be a selfjoining of $\mathcal{X}$ with $\vartheta(F) = 1$. Then clearly $\vartheta \not\gg \mu \times \mu$ and (iii) with $\mathcal{Y} = \mathcal{X}$ implies that μ-a.e. μ_x is discrete where $\vartheta = \int \delta_x \times \mu_x d\mu(x)$, i.e., F is sparse. □

Consider the following property of $\mathcal{X}$.

$$\text{(iii)}' \quad \begin{cases} \text{For every ergodic } \mathcal{Y}=(Y,\mathcal{F},\eta,T) \text{ and every \textit{ergodic} joining} \\ \vartheta \in \mathcal{P}(X\times Y),\ \vartheta \neq \mu\times\eta, \text{ there exists an integer} \\ n_0 \geqq 1 \text{ such that for } \eta\text{-a.e. } y, \\ \mu_y \text{ is supported on } n_0 \text{ points where } \vartheta = \int \mu_y \times \delta_y d\eta(y) \\ \text{(i.e., } (X\times Y,\nu)\to(Y,\eta) \text{ is } n_0 \text{ to one } \eta\text{-a.e.).} \end{cases}$$

By Proposition 2.1, (iii) $\Rightarrow$ (iii)$'$. By a result of M. Ratner the horocycle transformation [6] $\mathcal{X}=(G/\Gamma,\mu,h_1)$ has property (iii)$'$. However, since the set $\{(x,h_tx): x\in G/\Gamma, t\in\mathbf{R}\}=F_1$ is a subset of the set F of nongeneric points for $\mu\times\mu$, and since for every probability continuous measure ν on $\mathbf{R}$ the measure $\varphi_*(\nu\times\mu)\in\mathcal{P}(G/\Gamma\times G/\Gamma)$ (where $\varphi:\mathbf{R}\times G/\Gamma\to G/\Gamma$ is given by $\varphi(t,x)=(x,h_tx)$) is an $h_1\times h_1$-invariant continuous measure supported on F_1, we see that $\mathcal{X}$ is not an M-process. Thus (iii)$'$ $\not\Rightarrow$ (iii).

COROLLARY 2.3. *Let $\mathcal{X}$ be an M-process.*

(1) *If $\mathcal{X}\xrightarrow{\varphi}\mathcal{Y}$ is a factor, then φ is* a.s. *n to* 1. *In particular, $\mathcal{X}$ has a nontrivial prime factor.*

(2) *There are only countably many ergodic selfjoinings on $\mathcal{X}\times\mathcal{X}$.*

(3) *There are only countably many measure-preserving transformations, S on X with $ST=TS$.*

PROOF. (1) Follows from (iii)$'$ and Proposition 1.4. (2) Let $\mathcal{J}$ be the convex compact set of $T\times T$ invariant measures on $X\times X$ with marginals μ. Then the set $\mathcal{E}$ of ergodic joinings on $\mathcal{X}\times\mathcal{X}$ is the set of extreme points of $\mathcal{J}$. If the latter is uncountable, then $\mathcal{J}$ must contain measures ϑ for which (iii) cannot be satisfied. (3) This is a consequence of (2) since for every such S the measure $(I\times X)\mu_\Delta$ is an ergodic selfjoining on $\mathcal{X}\times\mathcal{X}$. □

Consider $\mathcal{X}=(X,\mathcal{B},\mu,T)$ an ergodic topological process such that (a) $Z(T)=\{S: S$ is a measure-preserving transformation of X with $ST=TS\}$ is countable and (b) every ergodic measure ϑ on X^n (for every n) with marginals μ, is (after a change of coordinates) of the form

$$\vartheta=\mu^k\times\prod_{i=1}^{l}(I\times S_{i,1}\times\cdots\times S_{i,n_i})(\mu_{\Delta_{n_i}})$$

where $k+\Sigma_{i=1}^{l} n_i=n$, $S_{i,j}\in Z(T)$ and $\mu_{\Delta_{n_i}}$ is the diagonal measure on X^{n_i}.

Call a process $\mathcal{X}$ which has such a topological realization *primitive*. Clearly every process with minimal selfjoinings in the sense of [7] is primitive. We now have the following theorem which was proved first by D. Rudolph and del Junco [5].

THEOREM 2.4. *Let $\mathscr{X}$ be primitive; then $\mathscr{X}$ is an M-process, and for every ergodic process $\mathscr{Y}$, either $\mathscr{X}$ is disjoint from $\mathscr{Y}$ or $\mathscr{Y}$ admits the process $(X^{\hat{n}}, \mu^{\hat{n}})$ as a factor for some $n \geqq 1$. (Here $\mu^{\hat{n}}$ is the image of $\mu \times \cdots \times \mu$ in $X^{\hat{n}}$.)*

PROOF. The fact that $\mathscr{X}$ is an M-process follows immediately from the definitions of primitive and M-processes. Now use Theorem 2.2 and the remark which follows it to conclude that $\mathscr{X}$ has property (iii)'. So let $\mathscr{Y} = (Y, \mathscr{F}, \eta, T)$ be ergodic and not disjoint from $\mathscr{X}$, and let $\vartheta \in \mathscr{P}(X \times Y)$ be an ergodic joining $\neq \mu \times \eta$. Write $\vartheta = \int \mu_y \times \delta_y d\eta(y)$ and let $\psi : Y \to \mathscr{P}(X) : \psi(y) = \mu_y$, $\lambda = \psi_*(\eta)$. Then $\mathscr{Y} \xrightarrow{\psi} (\mathscr{P}(X), \lambda, T)$ is a homomorphism, and by property (iii)' and Proposition 2.1, $(\mathscr{P}(X), \lambda, T)$ is isomorphic to a process $(X^{\hat{n}}, \sigma, T)$ with $(\pi_i)_*(\tilde{\sigma}) = \mu$, $1 \leqq i \leqq n$. (Recall that $\tilde{\sigma}$ is the unique lift of σ to X^n invariant under S_n.) Let σ_0 be an ergodic component of $\tilde{\sigma}$; then by primitivity σ_0 is a product of the form $\mu^k \times \Pi_{i=1}^l \sigma_i$, where $\sigma_1 = (I \times S_{i,1} \times \cdots \times S_{i,n_i})\mu_\Delta$. If $k = n$, then $\sigma = \mu^{\hat{n}}$; otherwise, since σ_i is isomorphic to μ, we actually have $\mathscr{X}$ itself as a factor of $\mathscr{Y}$. □

I would like to thank H. Furstenberg and B. Weiss for helpful suggestions and conversations. The history of the results in the second section is somewhat involved. In an attempt to prove Theorem 2.4 for the Chacon transformation, I formulated a variant of the definition of an M-process. In fact, I showed A. del Junco and M. Kean how Theorem 2.4 will follow for every process $\mathscr{X}$ for which the set F of nongeneric points in $X \times X$ is such that $\forall x$ F_x is countable, and asked them whether this is the case for the Chacon transformation. They succeeded in proving it for this transformation (thus showing that it satisfies the conclusion of Theorem 2.4). However, D. Rudolph then showed that Theorem 2.4 holds for any process with minimal selfjoinings. I realized later on that a weakening of my original definition of property M, the present one will still suffice for proving Theorem 2.4 and is actually equivalent to having only discretely supported q.f. (Theorem 2.2). I wish to thank also A. del Junco and M. Keane. Their result about the Chacon transformation has not been published.

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